Explicit Construction of First Integrals with Quasi-monomial Terms from the Painlevé Series

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Abstract The Painlevé and weak Painlevé conjectures have been used widely to identify new integrable nonlinear dynamical systems. For a system which passes the Painlevé test, the calculation of the integrals relies on a variety of methods which are independent from Painlevé analysis. The present paper proposes an explicit algorithm to build first integrals of a dynamical system, expressed as 'quasi-polynomial' functions, from the information provided solely by the Painlevé - Laurent series solutions of a system of ODEs. Restrictions on the number and form of quasi-monomial terms appearing in a quasi-polynomial integral are obtained by an application of a theorem by Yoshida (1983). The integrals are obtained by a proper balancing of the coefficients in a quasi-polynomial function selected as initial ansatz for the integral, so that all dependence on powers of the time $\tau = t - t_0$ is eliminated. Both right and left Painlevé series are useful in the method. Alternatively, the method can be used to show the non-existence of a quasi-polynomial first integral. Examples from specific dynamical systems are given.

1 Introduction

The present paper deals with autonomous dynamical systems described by ordinary differential equations of the form

$$\dot{x_i} = F_i(x_1, x_2, ..., x_n), \qquad i = 1, 2, ..., n. \tag{1}$$

where the functions F_i are of the form

$$F_i = \sum_{j=1}^{N} a_{ij} x^{Q_{ij}} \tag{2}$$

with $x \equiv (x_1, x_2 \dots x_n)$, $Q_{ij} \equiv (q_{ij1}, q_{ij2}, \dots, q_{ijn})$, $x^{Q_{ij}} \equiv \prod_{k=1}^{n} x_k^{q_{ijk}}$, and the exponents q_{ijk} are assumed to be rational numbers, i.e., the r.h.s. of Eq.(2) is a sum of quasimonomials (Goriely 1992).

A question of particular interest concerns the existence of *first integrals* of the system of ordinary differential equations (1). A first integral is a function $I(x_i)$ satisfying the equality $dI/dt = \nabla_x I \cdot \dot{x}(t) = 0$, where $x(t) \equiv (x_1(t), \dots, x_n(t)$ is any possible solution of (1). First integrals are important because they allow one to constrain the orbits on manifolds of dimensionality lower than n. In particular, a system (1) is called completely integrable if it admits $n-1$ independent and single-valued first integrals. Integrable systems exhibit regular dynamics, while the lack of a sufficient number of first integrals very often results in complex, chaotic dynamics.

The question of the existence of an algorithmic method which can determine all the first integrals of (1) is an important open problem in the theory of ordinary differential equations. Most relevant to this question are the methods of a) direct search or method of undetermined coefficients (e.g. Hietarinta 1983, 1987) b) normal forms and formal integrals (see Arnold 1985, Haller 1999 and Gorielly (2001) for a review), and c) Lie group symmetry methods (e.g. Lakshmanan and Senthil Velan (1992a,b), Marcelli and Nucci (2003)).

Even more difficult is the question of an algorithmic method probing the integrability of Eqs.(1). The most relevant method here is *singularity analysis*. According to the 'Painlevé conjecture' (Ablowitz, Ramani and Segur 1980), a system possessing the Painlevé property should be integrable. The Painlevé property means that any global solution of (1) in the complex time plane should be free of movable critical points other than poles. According to the 'weak-Painlevé conjecture (Ramani et al. 1982, Grammaticos et al. 1984, Abenda et al. 2001), however, certain types of movable branch points are compatible with integrability. Algorithms providing necessary conditions for a system to be Painlevé are a) the classical ARS test (Ablowitz et al. 1980) b) the perturbative Painlevé - Fuchs test (Fordy and Pickering 1991) which examines the role of negative resonances, and c) the generalized Painlevé test of Goriely (1992) which introduces coordinate transformations clarifying the nature of the singularity structure in the complex time domain. On the other hand, there is no algorithm, to the present, determining sufficient conditions for a system to be Painlevé. The most important obstacle is the search for essential singularities, which are not detectable by any of the above Painlevé tests.

The present paper explores the following question: Is it possible to recover the first integrals of a system by the information provided solely by singularity analysis of its differential equations? Our answer is partially affirmative. We cannot circumvent the difficulty concerning the choice of initial ansatz for the functional form of the integral. The most natural choice for quasi-polynomial equations (2) is to consider also a quasi-polynomial ansatz for the integral, with undetermined quasi-monomial coefficients.

This freedom in the initial ansatz notwithstanding, we show in the present paper that singularity analysis gives indeed the remaining information needed to recover the integrals.

First, as proposed by Roekaerts and Schwarz (1987), a theorem by Yoshida (1983) on the relation between Kowalevski exponents and weights of weighted-homogeneous integrals, can be used to impose restrictions on the degree of the quasi-monomial terms in the integral by analysing the resonances found by the Painlevé method. Now, Yoshida's theorem for weighted-homogeneous integrals is applicable on two conditions: If $b_i\tau^{-\lambda_i}$ is a balance of the system ($\tau = t - t_0$ is the time around the singularity t_0), and I is a weighted homogeneous integral, the theorem holds if a) $\nabla I(b_i)$ is finite, and b) $\nabla I(b_i) \neq 0$. These conditions shall be refered to as 'Yoshida's conditions'. The latter impose a severe restriction in the search for first integrals, because one cannot specify in advance whether these conditions are satisfied until the integrals are determined. In conclusion, the theorem of Yoshida has only indicative power as regards restrictions on the degree of quasimonomial terms in the integral. On the other hand, an analysis of the integrable Hamiltonian systems of two degrees of freedom with a polynomial potential given by Hiterinta (1983) shows that they all satisfy Yoshida's condition when the balance is taken equal to the 'principal' balance, in which all the b_i are different from zero. Yet, the extent of applicability of this result to other types of systems is unknown. In fact, we were able to find also a counterexample concerning a Hamiltonian proposed by Holt (1982).

Assuming a quasi-polynomial functional form of the integral, say $\Phi(x; c_1, c_2, ..., c_M)$,as

above, with undetermined quasi-monomial coeficients $c_i, i = 1, \ldots M$, the question now is whether it is possible to determine the coefficients c_i by the series derived via singularity analysis. The answer to this question is affirmative. Namely, by the usual Painlevé tests, the Painlevé-type series solutions around movable singularities are first identified

$$x_i(\tau) = \frac{1}{\tau^{\lambda_i}} \sum_{m=0}^{\infty} b_m \tau^m, \quad i = 1, \ldots n \tag{3}$$

Then, the series (3) *is substituted* into the quasi-polynomial function $\Phi(x; c_1, c_2, ..., c_M)$. The resulting expression is a Puisseux series, i.e., a series in rational powers of τ

$$\Phi(x_1(\tau), \ldots, x_n(\tau); c_1, c_2, ..., c_M) = \tau^{q/p} \sum_{m=0}^{\infty} d_m(b_i; c_i) \tau^{m/p} \tag{4}$$

with q, p, m integers. The coefficients $d_m(b_i; c_i)$ are nonlinear functions of the coefficients b_i (determined by (3), i.e., by singularity analysis), and linear functions of the undetermined coefficients c_i. But the function Φ is an integral of the system if it is constant along all the solutions of the system, including (3). As a result, the functions d_m satisfy the set of linear equations

$$d_m(b_i; c_i) = 0, \quad i = 0, 1, \ldots, \quad i \neq -q \tag{5}$$

This is an infinite number of homogeneous linear equations with a finite number of unknowns (the coefficients c_i). If M is the number of unknown coefficients, Eq.(5) can be written as

$$A(b_i) \cdot C = 0 \tag{6}$$

where $C = (c_1, \ldots, c_M)^T$ and $A(b_i)$ is a matrix with M columns and an infinite number of lines. The entries of A depend only on the coefficients b_i which were previously determined by singularity analysis. In this representation, the first integrals are functions with quasi-monomial coefficients given by the basis vectors of $ker(A)$ (or linear combinations of them). In computer algebraic implementations of the method, we work on a sub-matrix A_f defined by a finite number of lines, which is equal or larger than M. A basis for the subspace $ker(A_f)$ is determined by the singular value decomposition algorithm. Then, it is checked with direct differentiation that the resulting expression is an integral. This completes the determination of all quasi-polynomial first integrals for the given system.

This method was implemented in a number of examples presented below. Following some preliminary notions exposed in section 2, the results are presented in section 3, along with various details and implications in the implementation of the algorithm. Section 4 summarizes the main conclusions of the present study.

2 Preliminary notions

Following Yoshida (1983), a system of the form (1) is called scale-invariant if the equations remain invariant under the scale transformation $x_i \to a^{\lambda_i} x_i$, $t \to a^{-1} t$ for some λ_i. Then, the system (1) has exact special solutions of the form

$$x_i = \frac{b_i}{\tau^{\lambda_i}} \tag{7}$$

where $b_i, i = 1, \dots n$ is any of the sets of roots of the system of algebraic equations

$$F_i(b_1, b_2, ..., b_n) + \lambda_i b_i = 0$$

and the time $\tau = t - t_0$ is considered around any movable singularity t_0 in the complex time plane. Any solution of the form (7) is called a 'balance'. If the system (1) is scale-invariant and the functions F_i are of the form (2), the exponents λ_i associated with any of the balances are rational numbers. It should be stressed that the definition above does not require that all the b_i be non-zero. Balances of the form $x_i \sim 0/\tau^{\lambda_i}$, for some i, are also considered. The latter remark is essential in order to avoid the confusion which is sometimes made between 'balances' and 'dominant terms' in the ARS test (see e.g. the discussion between Steeb et al. (1987) and Ramani et al. (1988)), and the associated difference between 'Kowalevski exponents' and 'resonances'. In the standard ARS algorithm (Ablowitz et al. 1980) the solutions (7) arise by the definition of the dominant behaviors, i.e., which is the first step in the implementation of the algorithm.

The second step in the ARS algorithm is to look for series solutions that we call 'Painlevé series'. These are expansions of the form (3) starting with dominant terms of the form (7). They are Laurent (Taylor) series when the λ_i's are integers (positive integers), otherwise they can be series in rational powers of τ, which are called 'Puisseux series'. To build up the series, one first specifies the resonances, i.e. the values of r for which the coefficients of the terms $\tau^{r-\lambda_i}$ in the series are arbitrary. In the case when the coefficients of the balance b_i are all non-zero, the resonances are equal to the eigenvalues of the Kowalevski matrix $K_{ij} = (\partial F_i/\partial x_j + \delta_{ij}\lambda_i)|_{x_i=b_i}$ which are called 'Kowalevski exponents'. If, however, some of the b_is are equal to zero, then the resonances are not equal one to one to the Kowalevski exponents, but some resonances differ from the corresponding Kowalevski exponents by a quantity equal to the difference between the exponent λ_i in the balance and the exponent of the first non-zero dominant term in the Laurent-Puisseux series of $x_i(\tau)$, as specified in the first step of the ARS algorithm (Ramani et al. 1988).

At this point, we are *not* interested in whether the system passes the generalized Painlevé (or weak Painlevé) test. This means that we do *not* require that *all* the solutions of the system (1) can be written locally (around a movable singularity) in the form (3), or that there is at least one solution of the form (3) which contains n arbitrary constants (including t_0). On the other hand, we *do* check the compatibility conditions to ensure that no logarithms enter in the series. As regards positive resonances, compatibility is fulfilled automatically for scale-invariant systems. In summary, we are interested only that the system have special solutions of the form (3), but no other claim on it being Painlevé or not is required. Thus, the results are valid also for partially integrable systems, i.e., systems with a number of first integrals smaller than n.

Let us now assume that (1) possesses a weighted - homogeneous first integral Φ of weight M, i.e. an integral function Φ which satisfies the relation:

$$\Phi(a^{\lambda_1} x_1, a^{\lambda_2} x_2, ..., a^{\lambda_n} x_n) = a^M \Phi(x_1, x_2, ..., x_n) \tag{8}$$

for some M. Then we have the following

Theorem 1 (Yoshida 1983): *If, for a particular balance (7) the following conditions hold: a)$\nabla\Phi(b_i)$ is finite, and b) $\nabla\Phi(b_i) \neq 0$, then M is equal to one of the Kowalevski exponents associated with that balance.*

There has been a number of theorems in the literature linking Kowalevski exponents with the weights of homogeneous or quasi-homogeneous integrals of a system. Some characteristic papers on this subject are Llibre and Zhang (2002), Tsygvinsev (2001), Goriely (1996) and Furta (1996). However, the application of Yoshida's original theorem in the search for weighted-homogeneous integrals seems to be the most practical. Furthermore, Yoshida's theorem can be reformulated in an interesting way: consider the Painlené series starting with one of the balances (7). It follows that the series can be written as:

$$x_i = x_{iE} + R_i = \frac{b_i}{\tau^{\lambda_i}} + ... + A_i \tau^{r-\lambda_i} + R_i \tag{9}$$

where r is the maximum Kowalevski exponent associated with this balance and A_i is the corresponding arbitrary coefficient entering in the Painlevé series (9) for the variable x_i. The sum $x_{iE} = \frac{b_i}{\tau^{\lambda_i}} + ... + A_i \tau^{r-\lambda_i}$ will be called *essential part* of the series and the remaining part R_i *remainder* of the series. The remainder R_i starts with terms of degree $r - \lambda_i + 1/p$ where p is the denominator of λ_i written as a rational $\lambda_i = q/p$ with q, p coprime integers. A quasi-polynomial integral Φ has the form

$$\Phi = \sum c_{k1,k2,...,kn} \prod_{i=1}^{n} x_i^{k_i} \tag{10}$$

where the exponents k_i are rational numbers. If the integral Φ is weighted-homogeneous of weight M, we have

$$\sum_{i=1}^{n} k_i \lambda_i = M \tag{11}$$

due to (8). Taking into account the fact that the remainder R_i starts with terms of degree $O(\tau^{r-\lambda_i+1/p})$, it follows that the contribution of R_i in $x_i^{k_i}$ is in terms of degree $O(\tau^{-\lambda_i k_i + r + 1/p})$ or higher. This means that if the series (9) is substituted in the integral $\Phi(x_i)$, the contribution of R_i in $\Phi(x_i)$ is of degree $O(\tau^{-\sum \lambda_i k_i + r + 1/p}) = O(\tau^{-M+r+1/p})$ or higher. But $r \geq M$. Thus, the remainder R_i contributes only to terms of *positive* degree in τ. On the other hand, since Φ is an integral, the time τ as a denominator must be eliminated in $\Phi(x_i)$. But since there are no negative powers of τ generated in Φ by R_i, it follows that all the negative powers of τ are already eliminated by substituting the expression x_{iE} alone into Φ. Hence, we have the following

Proposition 2: *If the conditions of Yoshida's theorem hold for a weighted-homogeneous integral of (1) and a particular balance (7), then the expression $\Phi(x_{iE})$, where x_{iE} are the essential parts of Painlevé series $x_i(\tau)$ initiated with the same balance, does not contain singular terms in τ.*

Consider next the case when the functions F_i in (1) are not homogeneous. By the restrictions imposed by (2), it follows that the functions F_i can be decomposed in sums of the form

$$F_i = F_i^{(m_{i0})} + F_i^{(m_{i0}+1/p)} + ... + F_i^{(m_{i0}+q/p)} \tag{12}$$

where the functions $F_i^{(j)}$ are homogeneous of degree j, with j, m_{i0} rational, p, q integer, and p is the denominator in the simplest fraction giving m_{i0}. In this case, if the system (1) has

Painlevé type solutions, the dominant behaviors and associated resonances of these solutions are determined by the homogeneous term of the highest degree $F_i^{(m_{i0}+q/p)}$. On the other hand, the functions $F_i^{(j)}$, $j < m_{i0} + q/p$ must have a special form to ensure that compatibility conditions are fulfilled and the series solution is of the Painlevé type. Finally, as regards potential first integrals, the assumption that they consist of a sum of quasi-monomial terms implies that they can also be written as sums of the form (12). The selection of terms in the quasi-polynomial integral can be determined by Yoshida's theorem (or proposition 2) implemented in the scale-invariant systems

$$\dot{x}_i = F_i^{(m_{i0})}(x_i) \tag{13}$$

and

$$\dot{x}_i = F_i^{(m_{i0}+q/p)}(x_i) \tag{14}$$

respectively (Nakagawa 2002).

3 Explicit construction of integrals with quasi-monomial terms

3.1 An elementary example

Consider the two-dimensional nonlinear system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - 3x_1^2 \tag{15}$$

The only first integral of this system

$$\Phi = x_1^2 + x_2^2 + 2x_1^3 \tag{16}$$

can be recovered by elementary means. However, we will use this example to illustrate the steps used by the present method. The corresponding homogeneous system containing the terms of maximum degree

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -3x_1^2 \tag{17}$$

has the unique balance $x_1 = -2/\tau^2$, $x_2 = 4/\tau^3$, with Kowalevski exponents (equal to resonances) -1 and 6. Combatibility conditions are fulfilled for the system (15), which admits the Laurent series solution

$$\begin{aligned} x_1(\tau) &= \frac{-2}{\tau^2} - \frac{1}{6} - \frac{1}{120}\tau^2 + a_4\tau^4 + O(\tau^6) \\ x_2(\tau) &= \frac{4}{\tau^3} - \frac{1}{60}\tau + 4a_4\tau^3 + O(\tau^5) \end{aligned} \tag{18}$$

where a_4 is an arbitrary parameter.

Following Yoshida's theorem, we shall look for an integral of the system (15) by requesting that this integral be a sum of weighted- homogeneous functions of weight not higher than $M = 6$. Since the equations (15) are polynomial, the integral will also be assumed polynomial. According to the definition of the weighted-homogeneous functions (8), the undetermined integral contains terms of the form $x_1^{q_1}x_2^{q_2}$ the exponents of which are restricted by the relation $2q_1 + 3q_2 \leq 6$. This leaves only six possibilities, namely $(q_1 = 1, q_2 = 0)$, $(q_1 = 0, q_2 = 1)$,

$(q_1 = 2, q_2 = 0)$, $(q_1 = 1, q_2 = 1)$, $(q_1 = 0, q_2 = 2)$, $(q_1 = 3, q_2 = 0)$. Thus the integral is assumed to have the form

$$\Phi = c_{10}x_1 + c_{01}x_2 + c_{20}x_1^2 + c_{11}x_1x_2 + c_{02}x_2^2 + c_{30}x_1^3 \tag{19}$$

Up to now, the steps are exactly as proposed by Roekaerts and Schwarz (1987). At this point, however, we do not proceed by the 'direct method'; instead, the series (18) is substituted into (19). Then, terms of equal power in τ are separated and their respective coefficients are set equal to zero. We must determine at least as many equations as the number of unknown coefficients c_{ij}, i.e., six equations. These are:

$$\begin{aligned}
&Order\ \ O(1/\tau^6): && 16c_{02} - 8c_{30} = 0\\
&Order\ \ O(1/\tau^5): && -8c_{11} = 0\\
&Order\ \ O(1/\tau^4): && 4c_{20} - 2c_{30} = 0\\
&Order\ \ O(1/\tau^3): && 4c_{01} - \frac{2}{3}c_{11} = 0\\
&Order\ \ O(1/\tau^2): && -2c_{10} + \frac{2}{3}c_{20} - \frac{2}{15}c_{02} - \frac{4}{15}c_{30} = 0\\
&Order\ \ O(1/\tau): && 0 = 0
\end{aligned}$$

These equations can be written in matrix form:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 16 & -8 \\ 0 & 0 & 0 & -8 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & -2 \\ 0 & 4 & 0 & -2/3 & 0 & 0 \\ -2 & 0 & 2/3 & 0 & -2/15 & -4/15 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_{10} \\ c_{01} \\ c_{20} \\ c_{11} \\ c_{02} \\ c_{30} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{20}$$

or simply

$$A_f \cdot C = 0 \tag{21}$$

where C is a six-dimensional vector and A_f is a 6×6 matrix with constant entries.

The singular value decomposition of A_f yields a one-dimensional null space:

$$ker(A_f) = \lambda(0, 0, 1, 0, 1, 2, 0) \tag{22}$$

The basis vector $(0, 0, 1, 0, 1, 2)$ corresponds to the first integral $\Phi = x_1^2 + x_2^2 + 2x_1^3$.

A few remarks are here in order:

a) the last line of A_f has only zero entries, since it corresponds to the identity $0 = 0$ for the $O(1/\tau)$ terms. This is not a problem, because lines with zero elements are allowed by the singular value decomposition algorithm which determines the subspace $ker(A_f)$.

b) The entries of A_f are constant numbers which depend only on the coefficients of the Laurent series (18). This is the crucial remark; it implies that the information on the first integral is contained in the Painlevé series built by singularity analysis.

c) The arbitrary parameter a_4 in (18) does not appear in A_f. This phenomenon is not generic. In general, all the arbitrary parameters of the Painlevé series appear in A_f. In the computer implementation of the algorithm, we proceed by giving fixed values to the arbitrary parameters. Although the choice of values affects the convergence of the Painlevé series, it does not influence the present algorithm which is based only in the formal properties of the series.

3.2 Further examples

Of particular interest in nonlinear dynamics are autonomous Hamiltonian systems of two degrees of freedom of the form

$$H \equiv \frac{1}{2}(p_x^2 + p_y^2) + V(x,y) \tag{23}$$

where $V(x,y)$ is of the form (2). The easiest examples are systems with a polynomial potential (e.g. Hietarinta 1983, 1987). For example:

$$H \equiv \frac{1}{2}(p_x^2 + p_y^2 + x^2 + y^2) - x^2y - 2y^3 \tag{24}$$

This system passes the Painlevé test and it is integrable (Bountis et al., 1982). Keeping only the highest order terms of the potential $(-x^2y - 2y^3)$ yields the principal balance $x = 6i/\tau^2, y = 3/\tau^2$ with resonances $r = -3, -1, 6, 8$, which are equal to the corresponding Kowalevski exponents. The Painlevé series generated by the Hamiltonian (24) and the above principal balance satisfies the compatibility conditions of the ARS test. Assuming now a polynomial first integral Φ of (24), only the monomial terms of weight less or equal to 8 will be included to it. Since the leading terms of the momenta are $p_x \sim p_y \sim O(1/\tau^3)$, the selected monomial terms are:

$$x^4,\ x^3y,\ x^2y^2,\ xy^3,\ y^4,\ p_x^2x,\ p_xp_yx,\ p_y^2x,\ p_x^2y,\ p_xp_yy,\ p_y^2y \quad \text{(weight 8)}$$

$$p_xx^2,\ p_xxy,\ p_xy^2,\ p_yx^2,\ p_yxy,\ p_yy^2, \quad \text{(weight 7)}$$

$$x^3,\ x^2y,\ xy^2,\ y^3,\ p_x^2,\ p_xp_y,\ p_y^2, x^3,\ x^2y,\ xy^2\ , y^3 \quad \text{(weight 6)}$$

$$p_xx,\ p_xy,\ p_yx,\ p_yy, \quad \text{(weight 5)}$$

$$x^2,\ xy,\ y^2,\ p_x,\ p_y,\ x,\ y \quad \text{(weights 4,3,2)}$$

It should be stressed that there are several other restrictions that reduce the number of eligible terms. For example, the integral Φ is either even or odd in the momenta (Nakagawa and Yoshida 2001). Furthermore, the linear terms can be omitted by appropriate transformations. However, in the practical implementation of the algorithm these restrictions only introduce a complication, because, if an integral Φ exists, the singular value decomposition algorithm selects the basis for the corresponding null space of the matrix A_f without needing any extra information on restrictions which are specific for the system under study, e.g. hamiltonian or other.

Following the selection of monomial terms, the algorithm proceeds in building the homogeneous system (6) as well as the finite restriction A_f of the matrix A. In this case, the dimension of A_f should be $M \times 38$, with $M \geq 38$, since there are 38 unknown coefficients of the above monomial terms. At this point, it does not matter which balance and Laurent-generated series are used to build the matrix A_f. In the above example, the principal balance leads to a 'special solution', since there are only two positive resonances ($r = 6, 8$) meaning that there are three arbitrary parameters in total entering in the series. On the other hand, the general Laurent series solution with four arbitrary constants is given by a different

balance, namely $x = 0/\tau^2, y = 1/\tau^2$ so that x starts with dominant terms of $O(1/\tau)$, i.e.

$$
\begin{aligned}
x(\tau) &= \frac{A}{\tau} + a_1\tau + B\tau^2 + a_3\tau^3 + a_4\tau^4 a_5\tau^5 + O(\tau^6) \\
y(\tau) &= \frac{1}{\tau^2} + b_0 + b_2\tau^2 + b_3\tau^3 + C\tau^4 + O(\tau^5) \\
p_x(\tau) &= -\frac{A}{\tau^2} + a_1 + 2B\tau + 3a_3\tau^2 + 4a_4\tau^3 5a_5\tau^4 + O(\tau^5) \\
y(\tau) &= -\frac{2}{\tau^3} + 2b_2\tau + 3b_3\tau^2 + 4C\tau^3 + O(\tau^4)
\end{aligned} \tag{25}
$$

where A, B, C together with $t_0 = t - \tau$ are arbitrary, and $b_0 = (1 - A^2)/12$, $a_1 = A(1 - 2b_0)/2$, $b_2 = (b_0 - 6b_0^2 - 2Aa_1)/10$, $a_3 = (2b_0a_1 - a_1 + 2Ab_2)/4$, $b_3 = -AB/3$, $a_4 = (2Ab_3 - B - 2Bb_0)/10$, $a_5 = (2AC + 2a_1b_2 + 2b_0a_3 - a_3)/18$. In this solution, the resonances $r = -1, 0, 3, 6$ do not coincide one by one to the Kowalevski exponents $(-1, 1, 4, 6)$ deduced by the Kowalevski matrix associated with the above balance. Nevertheless, the general solution (25), as well as any other solution work equally well in determining the matrix A_f. In our case, by performing the singular value decomposition of A_f, the subspace $ker(A_f)$ yields two independent integrals in involution, with coefficients (up to the computer precision)

$$
\begin{aligned}
\Phi_1 = \; & 0.211944988455(y^2 + p_y^2 - 4y^3) + 0.049612730813(x^2 + p_x^2) \\
& +0.216443010189(xp_xp_y - p_x^2 y - x^2y^2 - \frac{1}{4}x^4) \\
& -0.207446966722x^2y
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
\Phi_2 = \; & 0.035960121414(y^2 + p_y^2 - 4y^3) + 0.342416478352(x^2 + p_x^2) \\
& -0.408608475918(xp_xp_y - p_x^2 y - x^2y^2 - \frac{1}{4}x^4) \\
& -0.480528718746x^2y
\end{aligned} \tag{27}
$$

in terms of which we can express the Hamiltonian

$$H = 2.16456459362950\Phi_1 + 1.146586289822\Phi_2 \tag{28}$$

and find a second integral orthogonal to the hamiltonian

$$I_2 = -2.16456459362950\Phi_2 + 1.146586289822\Phi_1 \tag{29}$$

The integral I_2 is a linear combination of the hamiltonian and of the integral I_b given by Bountis et al. (1982)

$$I_2 = 0.1651752123636227(2H - \frac{12}{7}I_b) \tag{30}$$

Let us note that an alternative way to obtain the matrix A_f is by considering only the essential parts of the Laurent series (25), for as many different sets of values of the arbitrary parameters as requested in order to have a complete determination of the $M \times 38$ elements of A_f, with $M \geq 38$. This approach is preferable in computer implementations of the algorithm, because one does not need to calculate the terms of the Laurent series (25) beyond the highest positive resonance. Furthermore, Proposition 2, instead of Yoshida's theorem, can be used to select the quasiminomial basis set.

A final remark concerns the applicability of Yoshida's condition $\nabla\Phi(b_i) \neq 0$. We checked whether this condition is satisfied in Hientarinta's table (1983) of integrable Hamiltonian systems of two degrees of freedom with a polynomial homogeneous potential. There are six non-trivial cases:

$$\text{(a)} \quad V(x,y) = 2x^3 + xy^2$$

with integral $\Phi = yp_xp_y - xp_y^2 + x^2x_2 + \frac{1}{4}y^4$

$$\text{(b)} \quad V(x,y) = \frac{16}{3}x^3 + xy^2$$

with integral $\Phi = p_y^4 + 4xy^2p_y^2 - \frac{4}{3}y^3p_xp_y - \frac{4}{3}x^2y^4 - \frac{2}{9}y^6$

$$\text{(c)} \quad V(x,y) = 2x^3 + xy^2 + i\frac{\sqrt{3}}{9}y^3$$

with integral $\Phi = p_y^4 + \frac{2}{\sqrt{3}}ip_xp_y^3 + \frac{2}{\sqrt{3}}iy^3p_x^2 - (2y^3 + 2i\sqrt{3}xy^2)p_xp_y + (4i\sqrt{3}x^2y + 2i\sqrt{3}y^3 + 4xy^2)p_y^2 + \frac{4}{\sqrt{3}}ix^3y^3 + \frac{2}{\sqrt{3}}ixy^5 - x^2y^4 - \frac{5}{9}y^6$

$$\text{(d)} \quad V(x,y) = \frac{4}{3}x^4 + x^2y^2 + \frac{1}{12}y^4$$

with integral $\Phi = yp_xp_y - xp_y^2 + (\frac{1}{3}2x^3 + xy^2)y^2$

$$\text{(e)} \quad V(x,y) = \frac{4}{3}x^4 + x^2y^2 + \frac{1}{6}y^4$$

with integral $\Phi = p_y^4 + \frac{2}{3}y^4p_x^2 - \frac{8}{3}xy^3p_xp_y + (4x^2y^2 + \frac{2}{3}y^4)p_y^2 + \frac{1}{9}(y^8 + 4x^2y6 + 4x^4y^4)$ and

$$\text{(f)} \quad V(x,y) = x^5 + x^3y^2 + \frac{3}{16}xy^4$$

with integral $\Phi = yp_xp_y - xp_y^2 + \frac{1}{2}x^4y^2 + \frac{3}{8}x^2y^4 + \frac{1}{32}y^6$.

In all these cases, the principal balance, with all b_i different from zero, satisfies Yoshida's condition. Note that case (f) the principal balance leads to 'weak-Painlevé' solutions, but the associated integral Φ is easily recoverable by the present algorithm. In fact, the role of Yoshida's condition is to exclude from Theorem 1 integrals which are composite functions of a simpler integral. For example, consider the integral $I = \Phi^2$, where Φ is the polynomial integral in any of the above six Hamiltonian systems. In all of them Φ is weighted-homogeneous for some weight M. Thus I is weighted-homogeneous of weight $2M$. Substituting the special solution (7) in the integral Φ yields $\Phi(b_i/\tau^{\lambda_i}) = \tau^{-M}\Phi(b_i)$. Since Φ is an integral, it should be time-independent, thus $\Phi(b_i) = 0$. Similarily, $I(b_i) = 0$. However, while $\nabla\Phi(b_i) \neq 0$, we have $\nabla I = 2\Phi\nabla\Phi$, thus $\nabla I(b_i) = 0$. Thus, while Φ satisfies the Yoshida's condition, I does not. This ensures that while M is necessarily a Kowalevski exponent, the multiples of it, e.g. $2M$ are not necessarily Kowalevski exponents. Viewed under this context, it appears that the condition $\nabla\Phi(b_i) \neq 0$ makes a 'natural' choice of the simplest integral among an infinity of possible integrals which are composite functions of Φ.

However, there are interesting counterexamples which challenge this point of view. One example is the homogeneous limit of the Holt (1982) Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) - \frac{3}{4}x^{4/3} - y^2x^{-2/3} \tag{31}$$

In this case, the potential is a homogeneous sum of quasi-monomials, of degree 4/3. The second integral reads:

$$\Phi = 2\dot{y}^3 + 3\dot{y}\dot{x}^2 - 6\dot{y}y^2x^{-2/3} + 9\dot{y}x^{4/3} - 18\dot{x}yx^{1/3} \tag{32}$$

and it is weighted-homogeneous of degree $M = -6$. The principal balance is

$$x = \left(\frac{1}{3}\right)^{3/2}\tau^3, \quad y = \left(\frac{i}{3\sqrt{2}}\right)\tau^3 \tag{33}$$

with resonances (=Kowalevski exponents) $r = -1, -2, -3, -4$. This case is remarkable because a) all the resonances are negative, and b) they are not equal to the weight of the integral $M = -6$. Thus, Yoshida's theorem does not apply in the case of this balance. Substituting the balance $b = ((1/3)^{3/2}, 1/3\sqrt{2}, 3(1/3)^{3/2}, 1/\sqrt{2})$, i.e., Eq.(33) in the gradient of the integral (32),

$$\begin{aligned} \nabla\Phi(x, y, \dot{x}, \dot{y}) \;=\; & (4\dot{y}y^2x^{-5/3} + 12\dot{y}x^{1/3} - 6\dot{x}yx^{-2/3}, \\ & -12\dot{y}yx^{-2/3} - 18\dot{x}x^{1/3}, 6\dot{x}\dot{y} - 18yx^{1/3}, 6\dot{y}^2 + 3\dot{x}^2 - 6y^2x^{-2/3} + 9x^{4/3}) \end{aligned} \tag{34}$$

yields $\nabla\Phi(b) = 0$. Thus none of the resonances has to be equal to the weight of the integral. However, the form of Φ or $\nabla\Phi$ does not suggest that these functions are composite functions of some simpler integral. On the other hand, there is a second balance of the Hamiltonian (31), namely $b = ((1/6)^{3/2}, 0, 3(1/6)^{3/2}, 0)$, which corresponds to the dominant behavior

$$x = \left(\frac{1}{6}\right)^{3/2}\tau^3, \quad y = A\tau^4 \tag{35}$$

with A arbitrary. The resonances here are $r = 0, -1, -4$ and -7, but the Kowalevski exponents are $r_K = 1, -1, -4, -6$ (two of them differ by one from the respective resonances). Now, the Kowalevski exponent $r_k = -6$ is equal to the weight of the integral. If we look at $\nabla\Phi$, Eq.(34), we see that the components $\partial\Phi/\partial y$ and $\partial\Phi/\partial\dot{y}$ contain terms independent of y and $\dot{y}$. Thus, for this particular balance $\nabla\Phi(b) \neq 0$, i.e., Yoshida's condition is satisfied.

Thus, in the absence of a counterexample, we formulate the following

Conjecture 3: *In any scale-invariant system of the form (1,2), which possesses a weighted-homogeneous first integral Φ, at least one of the balances b satisfies the condition $\nabla\Phi(b) \neq 0$.*

Returning to the Holt Hamiltonian, the next step is the selection of quasi-monomial terms in the initial ansatz for a quasi-polynomial integral. Guided by the form of the Hamiltonian, natural exponents are adopted for the powers to which the momenta p_x, p_y, and of the variable y are raised, while the variable x is considered as raised to powers $m/3$, where m is integer (positive or negative). Even under these restrictions, there is an infinity of possible quasi-monomial terms of weight -6. For example, the terms $p_i x^{m/3} y^{(2-m)/3}$, where p_i is either p_x or p_y, are of weight -6 for all $m \in Z$. Thus an arbitrary lower limit has to be set to m. This is chosen as the lowest bound of m in the Hamiltonian, namely $m = -2$. Nevertheless, failure to find an integral with these restrictions on the quasi-monomial terms does not imply that an integral does not exist, because of the arbitrariness with respect to the lowest bound of negative exponents considered. This problem does not exist when there are only

positive exponents present in the quasi-monomial terms of the equations of motion (or of the Hamiltonian).

With the above restrictions, the quasi-monomial terms considered are:

$$p_x^3,\ p_x^2 p_y,\ p_x p_y^2,\ p_y^3,\ p_x^2 x^{2/3},\ p_x^2 x^{-1/3} y$$

$$p_x p_y x^{2/3},\ p_x p_y x^{-1/3} y,\ p_y^2 x^{2/3},\ p_y^2 x^{-1/3} y$$

$$p_x x^{4/3},\ p_x x^{1/3} y,\ p_x x^{-2/3} y^2,\ p_y x^{4/3}$$

$$p_y x^{1/3} y,\ p_y x^{-2/3} y^2,\ x^2, x^y, y^2$$

The final step is to build the matrix A_f as in the previous examples, i.e. by replacing the Painlevé series in the initial ansatz for the integral. An interesting point is that in the case of the Holt Hamiltonian (31) we have to consider *left Painlevé series* (Pickering 1996), i.e., series in descending powers of τ. This is because the balances $\sim \tau^3$ do not imply singular behavior as $\tau \to 0$. In this case, the limit $|\tau| \to \infty$ represents a singularity, but the left Painlevé series are convergent for all τ with $|\tau| > \epsilon$ for some real positive ϵ. The series are constructed as:

$$x_i(\tau) = \tau^\lambda \sum_{k=0}^{\infty} b_k \tau^{-k} \tag{36}$$

where $\lambda > 0$. Resonances and compatibilities are checked in the same way as in the usual Painlevé test. By this method, we were able to obtain the integral (32) by a proper balancing of the quasi-monomial coefficients c_i so as to eliminate the coefficients of the terms of successive descending powers of τ in the integral expression.

As a final remark, it should be stressed that the selection of a quasi-polynomial ansatz for the integrals of a system of the form (1), with the functions (2) being quasi-polynomial, is not exhaustive. This can be easily exemplified in a case with polynomial functions. The Bogoyavlensky - Volterra B-type systems are given in normalized coordinates u_i by the following set of autonomous nonlinear ODEs:

$$\begin{aligned} \dot{u_1} &= u_1^2 + u_1 u_2 \\ \dot{u_i} &= u_i u_{i+1} - u_i u_{i-1}, \qquad i = 2, ..., n-1 \\ \dot{u_n} &= -u_n u_{n-1} \end{aligned} \tag{37}$$

the r.h.s. of Eqs.(37) are homogeneous functions of second degree in the variables u_i. The system (37)) admits balances of the form of the form:

$$u_i = \frac{a_i}{\tau} \tag{38}$$

where $\tau = t - t_0$ is the time near a singularity t_0 in the complex t-plane. In the case of the principal balance, the a_i are non-zero solutions of the set of algebraic equations:

$$\begin{aligned} -a_1 &= a_1^2 + a_1 a_2 \\ -a_i &= a_i a_{i+1} - a_i a_{i-1}, \qquad i = 2, ..., n-1 \\ -a_n &= -a_n a_{n-1} \end{aligned} \tag{39}$$

and they are given by the recursion formulas

$$a_{k+2} = a_k - 1, \qquad a_1 = (-1)^n [\frac{n+1}{2}], \qquad a_2 = -1 - a_1 \tag{40}$$

for $k = 1, ..., n$.

The resonances of the principal balance (=Kowalevski exponents) are given by the characteristic equation, i.e., setting the determinant of the Kowalevski matrix equal to zero. The determinant has a tridiagonal form, i.e.,

$$det \begin{pmatrix} a_1 - r & a_1 & 0 & 0 & 0 & 0 & \dots & 0 \\ -a_2 & -r & a_2 & 0 & 0 & 0 & & 0 \\ 0 & -a_3 & -r & a_3 & 0 & 0 & & 0 \\ . & & . & . & . & 0 & & . \\ . & & & . & . & . & & . \\ . & & & & . & . & . & . \\ 0 & & & & & -a_{n-1} & -r & a_{n-1} \\ 0 & . & . & . & & 0 & -a_n & -r \end{pmatrix} = 0 \tag{41}$$

which can be solved easily yielding the resonances

$$r_k = (-1)^k k, \qquad k = 1, ..., n \tag{42}$$

Assuming that the conditions for Yoshida's theorem hold, the weight of a weighted-homogeneous integral of (37) should be one of the resonances (42). Indeed, we find an integral by singularity analysis for *any* of the positive resonances given by equation (42). The integrals are given by the recurrent relations:

$$I_n^{(m)} = I_{n-1}^{(m)} + u_n^2 I_{n-2}^{(m-2)} + 2 \sum_{k=0}^{m/2-1} [I_{n-m-1+2k}^{(2k)} \prod_{j=n-m+1+2k}^{n} u_j] \tag{43}$$

where the convention $I_n^{(0)} = 1$ and $I_n^{(m)} = 0$ for all m, n with $n = 2, 3, \dots$ and $m > n$ is adopted.

For $n = 3$, the resonances are $r = -3, -1, 2$ and a polynomial integral is

$$I_3^{(2)} = c^2 = u_2^2 + u_3^2 + 2u_1u_2 + 2u_2u_3 \tag{44}$$

However, it is simple to see that this is not the only first integral of the system (37). Defining $u = u_2 + u_3$, and using any constant value c of the integral (44), the equations of motion take the form

$$\dot{u} = \frac{1}{2}(u^2 - c^2) \tag{45}$$

Integration of (45) yields

$$u = -c \coth(\frac{c(t-t_0)}{2}) \tag{46}$$

Using u instead of u_2 as a new independent variable yields the equation:

$$\dot{u}_3 + uu_3 = u_3^2 \tag{47}$$

Which can be solved for u_3 yielding

$$u_3 = -c\frac{\cosh(c(t-t_0)) - 1}{\sinh(c(t-t_0)) - c(t-t_0) + 2c\gamma} \tag{48}$$

where γ is an integration constant. By eliminating the time between the solutions (46) and (48), a new first integral of the original equations is found:

$$I_{tr} = -2\gamma = \frac{1}{c}\ln(\frac{u_2 + u_3 - c}{u_2 + u_3 + c}) + \frac{2u_1 + u_2 + u_3}{u_1 u_3} \tag{49}$$

which is a transcendental function of the variables u_j. This integral could not have been found by the initial polynomial ansatz.

4 Conclusions

In this paper we have explored the question of whether it is possible to recover the first integrals of systems of first-order nonlinear ordinary different equations involving quasi-polynomial functions of the independent variables based on the information provided by singularity analysis. The main conclusions are:

a) The theorem of Yoshida (1983) constrains the choice of an initial ansatz for an integral with undetermined parameters, leaving, however, an infinity of possible choices.

b) The condition of Yoshida's theorem ($\nabla\Phi(b_i) \neq 0$ and finite) holds for all the integrable Hamiltonian systems of two degrees of freedom included in Hietarinta's (1983) table, if b_i is set equal to the principal balance $b_i \neq 0$ for all $i = 1, \ldots 4$. Other types of balances have to be considered in more general systems.

c) Substitution of the Painlevé series in a quasi-polynomial function $\Phi(x; c)$, where c is the vector of undetermined parameters, allows to separate the terms in powers of the time and determine the parameters c by singular value decomposition. Thus the information on quasi-polynomial integrals is contained in the Painlevé-type series solutions around any movable singularity.

d) In the case of balances τ^λ with $\lambda > 0$, left Painlevé series must be used in the implementation of the algorithm.

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